

This is a repository copy of *Note on the spectrum of classical and uniform exponents of Diophantine approximation*.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/130106/>

Version: Accepted Version

---

**Article:**

Marnat, Antoine (2018) Note on the spectrum of classical and uniform exponents of Diophantine approximation. *Acta Arithmetica*. ISSN 1730-6264

<https://doi.org/10.4064/aa170106-23-3>

---

**Reuse**

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

**Takedown**

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing [eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk) including the URL of the record and the reason for the withdrawal request.

# Note on the spectrum of classical and uniform exponents of Diophantine approximation

Antoine MARNAT\*  
[antoine.marnat@york.ac.uk](mailto:antoine.marnat@york.ac.uk)

## Abstract

Using the Parametric Geometry of Numbers introduced recently by W.M. Schmidt and L. Summerer [13, 14] and results by D. Roy [10, 11], we establish that the  $2n$  exponents of Diophantine approximation in dimension  $n \geq 3$  are algebraically independent.

## 1 Introduction

Throughout this paper, the integer  $n \geq 1$  denotes the dimension of the ambient space  $\mathbb{R}^n$  endowed with its Euclidean norm and  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  denotes an  $n$ -tuple of real numbers such that  $1, \theta_1, \dots, \theta_n$  are  $\mathbb{Q}$ -linearly independent.

Let  $d$  be an integer with  $0 \leq d \leq n-1$ . We define the exponent  $\omega_d(\boldsymbol{\theta})$  (resp. the uniform exponent  $\hat{\omega}_d(\boldsymbol{\theta})$ ) as the supremum of the real numbers  $\omega$  for which there exist rational affine subspaces  $L \subset \mathbb{R}^n$  such that

$$\dim(L) = d, \quad H(L) \leq H \quad \text{and} \quad H(L)d(\boldsymbol{\theta}, L) \leq H^{-\omega}$$

for arbitrarily large real numbers  $H$  (resp. for every sufficiently large real number  $H$ ). Here  $H(L)$  denotes the exponential height of  $L$  (see [12] for more details), and  $d(\boldsymbol{\theta}, L) = \min_{P \in L} d(\boldsymbol{\theta}, P)$  is the minimal distance between  $\boldsymbol{\theta}$  and a point of  $L$ . Note that this definition is independent of the choice of a norm on  $\mathbb{R}^n$ .

These exponents were introduced originally by M. Laurent [7]. They interpolate between the classical exponents  $\omega(\boldsymbol{\theta}) = \omega_{n-1}(\boldsymbol{\theta})$  and  $\lambda(\boldsymbol{\theta}) = \omega_0(\boldsymbol{\theta})$  (resp.  $\hat{\omega}(\boldsymbol{\theta}) = \hat{\omega}_{n-1}(\boldsymbol{\theta})$  and  $\hat{\lambda}(\boldsymbol{\theta}) = \hat{\omega}_0(\boldsymbol{\theta})$ ) that were introduced by A. Khintchine [4, 5], V. Jarník [3] and Y. Bugeaud

---

\*supported by the Austrian Science Fund (FWF), Project F5510-N26, and FWF START project Y-901 and EPSRC Programme Grant EP/J018260/1

and M. Laurent [1, 2].

We have the relations

$$\begin{aligned}\omega_0(\boldsymbol{\theta}) &\leq \omega_1(\boldsymbol{\theta}) \leq \cdots \leq \omega_{n-1}(\boldsymbol{\theta}), \\ \hat{\omega}_0(\boldsymbol{\theta}) &\leq \hat{\omega}_1(\boldsymbol{\theta}) \leq \cdots \leq \hat{\omega}_{n-1}(\boldsymbol{\theta}),\end{aligned}$$

and Minkowski's First Convex Body Theorem [9] and Mahler's compound convex bodies theory provide the lower bounds

$$\omega_d(\boldsymbol{\theta}) \geq \hat{\omega}_d(\boldsymbol{\theta}) \geq \frac{d+1}{n-d}, \text{ for } 0 \leq d \leq n-1.$$

These  $2n$  exponents happen to be related as was first noticed by Khinchin with his transference theorem [5]. We use the following notion of spectrum to study more general transfers. Given  $k$  exponents  $e_1, \dots, e_k$ , we define the *spectrum* of the exponents  $(e_1, \dots, e_k)$  as the subset of  $\mathbb{R}^k$  described by all  $k$ -uples  $(e_1(\boldsymbol{\theta}), \dots, e_k(\boldsymbol{\theta}))$  as  $\boldsymbol{\theta}$  runs through all points  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$  such that  $1, \theta_1, \dots, \theta_n$  are  $\mathbb{Q}$ -linearly independent.

In [8], the author proved the following theorem.

**Theorem 1.** *For every integer  $n \geq 3$ , the  $n$  uniform exponents  $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$  are algebraically independent.*

Using the same construction, it is even possible to show that for every integer  $n \geq 3$ , the spectrum of  $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$  is a subset of  $\mathbb{R}^n$  with non empty interior. In this paper, we extend this result as follows.

**Theorem 2.** *For every integer  $n \geq 3$ , the  $2n$  exponents  $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}, \omega_0, \dots, \omega_{n-1}$  are algebraically independent.*

In dimension  $n = 2$ , the spectrum is fully described in [6]:

**Theorem 3** (Laurent, 2009). *In dimension 2, the spectrum of the four exponents  $\omega_0, \omega_1, \hat{\omega}_0, \hat{\omega}_1$  is described by the inequalities*

$$\hat{\omega}_0 + 1/\hat{\omega}_1 = 1, \quad 2 \leq \hat{\omega}_1 \leq +\infty, \quad \frac{\omega_1(\hat{\omega}_1 - 1)}{\omega_1 + \hat{\omega}_1} \leq \omega_0 \leq \frac{\omega_1 - \hat{\omega}_1 + 1}{\hat{\omega}_1}.$$

*When  $\hat{\omega}_1 < \omega_1 = +\infty$  we have to understand these relations as  $\hat{\omega}_1 - 1 \leq \omega_0 \leq +\infty$  and when  $\hat{\omega}_1 = +\infty$ , the set of constraints should be interpreted as  $\omega_0 = \omega_1 = +\infty$  and  $\hat{\omega}_0 = 1$ .*

The first equality, relating the two uniform exponents, is known as Jarník's relation [3] and breaks the algebraic independence. Note that this sharpens previously mentioned relations. In dimension  $n = 1$  the uniform exponent is always equal to 1.

We refer the reader to [8, §2] for the notation and the presentation of the parametric geometry of numbers, main tool of the proof. We mainly use the notation introduced by D. Roy in [10, 11] which is essentially dual to the one of W. M. Schmidt and L. Summerer [13, 14].

## 2 Proof of the main Theorem 2

To prove Theorem 2, we place ourselves in the context of parametric geometry of numbers. We fully use Roy's theorem [8, Theorem 5] that reduces the study of spectra of Diophantine approximation to the study of the combinatorial properties of generalized  $n$ -systems. We construct explicitly a family of generalized  $(n + 1)$ -systems with  $2n$  parameters, which provides the algebraic independence in the spectrum via Roy's theorem.

We fix the dimension  $n \geq 3$ . Consider any family of positive parameters

$$A_1 = A_2 < A_3 < \cdots < A_{n+1}, B_2 < B_3 < \cdots < B_n, C, D$$

satisfying the following properties for  $2 \leq k \leq n$ :

$$\begin{aligned} A_1 + A_2 + \cdots + A_{n+1} &= 1, B_2 < D < CA_2, \\ A_{k+1} < B_k < A_{k+2}, B_k < CA_k, \end{aligned} \tag{1}$$

where  $A_{n+2} = \infty$ .

We consider the generalized  $(n + 1)$ -system  $\mathbf{P}$  on the interval  $[1, C]$  depending on the previous parameters whose combined graph is given below by Figure 1, where

$$P_k(1) = A_k \text{ and } P_k(C) = CA_k \text{ for } 1 \leq k \leq n + 1.$$

Conditions (1) are consistent with the graph. On each interval between two consecutive division points, there is only one line segment with non zero slope. This line segment has slope 1 on the intervals  $[1, \delta_{2,1}]$ ,  $[\delta_{k-1,2}, \delta_{k,1}]$  for  $3 \leq k \leq n$ , and  $[\mu_k, \mu_{k-1}]$  for  $n \geq k \geq 1$ , and has slope 1/2 on the interval  $[\mu_0, C]$  and  $[\delta_{k,1}, \delta_{k,2}]$  for  $3 \leq k \leq n$ , where the two components  $P_k$  and  $P_{k+1}$  coincide. We have  $3n + 1$  division points  $1, C, \delta_{k,1}$  and  $\delta_{k,2}$  for  $2 \leq k \leq n$  and  $\mu_l$  for  $n + 1 \geq l \geq 0$ . They are all ordinary division points except  $\mu_k$  for  $1 \leq k \leq n$  which are switch points.

The points which will be most relevant for the proof are labeled with black dots. Note that from 1 to  $\delta_{n,2}$ , the combined graph is the same as in [8, §5].

We extend  $\mathbf{P}$  to the interval  $[1, \infty)$  by self-similarity. This means,  $\mathbf{P}(q) = C^m \mathbf{P}(C^{-m}q)$  for all integers  $m$ . In view of the value of  $\mathbf{P}$  and its derivative at 1 and  $C$ , one sees that the extension provides a generalized  $(n + 1)$ -system on  $[1, \infty)$ .

The relation between exponents and  $n$ -systems [8, Proposition 1] suggests to define  $2n$

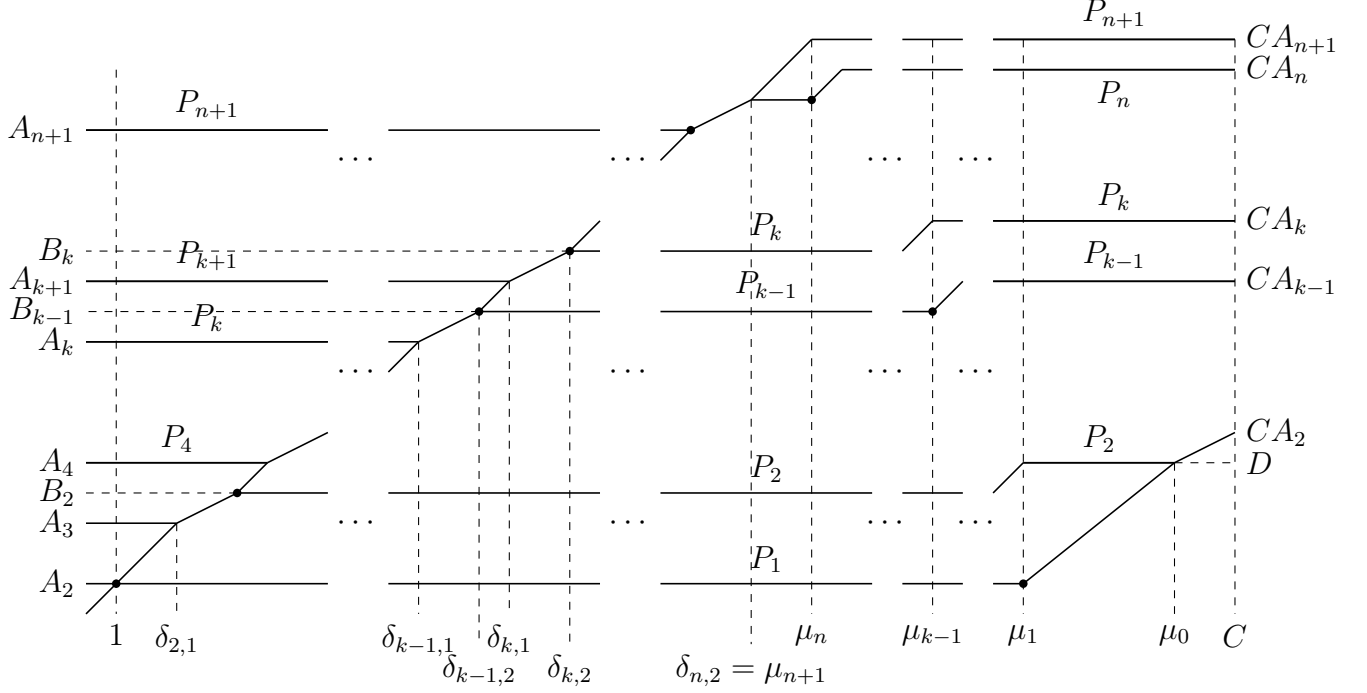


Figure 1: Pattern of the combined graph of  $\mathbf{P}$  on the fundamental interval  $[1, C]$

quantities  $W_{n-1}, \dots, W_0, \hat{W}_{n-1}, \dots, \hat{W}_0$  by

$$\frac{1}{1 + \hat{W}_{n-k}} := \limsup_{q \rightarrow +\infty} \frac{P_1(q) + \dots + P_k(q)}{q} \text{ for } 1 \leq k \leq n,$$

$$\frac{1}{1 + W_{n-k}} := \liminf_{q \rightarrow +\infty} \frac{P_1(q) + \dots + P_k(q)}{q} \text{ for } 1 \leq k \leq n.$$

Indeed with this setting, Roy's Theorem provides the existence of a point  $\boldsymbol{\theta}$  in  $\mathbb{R}^n$  such that  $\hat{\omega}_k(\boldsymbol{\theta}) = \hat{W}_k$  and  $\omega_k(\boldsymbol{\theta}) = W_k$  for every  $0 \leq k \leq n-1$ .

Here, self-similarity ensures that the lim sup (resp. lim inf) is in fact the maximum (resp. the minimum) on the interval  $[1, C[$ . Note that for  $1 \leq k \leq n$ , the function  $P_1 + \dots + P_k$  has slope 1 on the intervals  $[1, \delta_{k,1}]$  and  $[\mu_k, C[$ , slope 1/2 on the interval  $[\delta_{k,1}, \delta_{k,2}]$  and is constant on the interval  $[\delta_{k,2}, \mu_k]$ . Therefore the minimum of the function  $q \mapsto q^{-1}(P_1(q) + \dots + P_k(q))$  is reached at  $\mu_k$  and its maximum is reached either at  $\delta_{k,1}$  or at  $\delta_{k,2}$ , when slope changes from 1 to 1/2 or from 1/2 to 0. Namely, the maximum is reached at  $\delta_{k,1}$  if

$$\frac{P_1(\delta_{k,1}) + \dots + P_k(\delta_{k,1})}{\delta_{k,1}} \geq \frac{1}{2} \quad (2)$$

and at  $\delta_{k,2}$  if the lefthand side is  $\leq 1/2$ . We deduce that for  $1 \leq k \leq n$ ,

$$\begin{aligned}\hat{W}_{n-k} &= \frac{P_{k+1}(q) + \cdots + P_{n+1}(q)}{P_1(q) + \cdots + P_k(q)} \text{ where } q = \begin{cases} \delta_{k,1} & \text{if (2) is satisfied} \\ \delta_{k,2} & \text{otherwise} \end{cases}, \\ W_{n-k} &= \frac{P_{k+1}(\mu_k) + \cdots + P_{n+1}(\mu_k)}{P_1(\mu_k) + \cdots + P_k(\mu_k)}.\end{aligned}$$

It is easy to check that the parameters

$$\begin{aligned}C &= 3, A_1 = A_2 = 2^{-n}, A_k = 2^{-n+k-2} \text{ for } 3 \leq k \leq n+1 \\ D &= \frac{11}{8}2^{-n+1}, B_k = \frac{5}{4}2^{-n+k-1} \text{ for } 2 \leq k \leq n\end{aligned}\tag{3}$$

satisfy the conditions (1). For this choice of parameters, the lefthand side of inequality (2) is  $> 1/2$  for  $1 \leq k \leq n-1$  and  $< 1/2$  for  $k = n$ . This property remains true for  $(C, A_2, \dots, A_n, D, B_2, B_3, \dots, B_n)$  in an open neighborhood of the point

$$(3, 2^{-n}, \dots, 2^{-2}, \frac{11}{8}2^{-n+1}, \frac{5}{2}2^{-n}, \dots, \frac{5}{2}2^{-2})$$

provided that we set  $A_1 = A_2$  and  $A_{n+1} = 1 - (A_1 + \cdots + A_n)$ . In this neighborhood, the quantities  $W_0, \dots, W_{n-1}, \hat{W}_0, \dots, \hat{W}_{n-1}$  are given by the following rational fractions in  $\mathbb{Q}(C, A_2, \dots, A_n, D, B_2, B_3, \dots, B_n)$ :

$$\begin{aligned}\hat{W}_{n-1} &= \frac{1}{A_2} - 1, & \hat{W}_0 &= \frac{1 - (2A_2 + A_3 + A_4 + \cdots + A_n)}{A_2 + (B_2 + \cdots + B_{n-1})}, \\ \hat{W}_{n-k} &= \frac{1 - (2A_2 + A_3 + A_4 + \cdots + A_{k+1}) + B_k}{A_2 + (B_2 + \cdots + B_k)} \text{ for } 2 \leq k \leq n-1, \\ W_{n-k} &= \frac{C(1 - (2A_2 + A_3 + A_4 + \cdots + A_k))}{A_2 + B_2 + \cdots + B_k} \text{ for } 2 \leq k \leq n, \\ W_{n-1} &= \frac{D + C(1 - 2A_2)}{A_2}.\end{aligned}$$

Since  $W_0, \dots, W_{n-1}, \hat{W}_0, \dots, \hat{W}_{n-1}$  come from a generalized  $(n+1)$ -system  $\mathbf{P}$ , Roy's Theorem provides the existence of a point  $\boldsymbol{\theta}$  in  $\mathbb{R}^n$  such that  $\hat{\omega}_k(\boldsymbol{\theta}) = \hat{W}_k$  and  $\omega_k(\boldsymbol{\theta}) = W_k$  for every  $0 \leq k \leq n-1$ . Therefore, to prove Theorem 2 it is sufficient to show that the rational fractions  $W_0, \dots, W_{n-1}, \hat{W}_0, \dots, \hat{W}_{n-1} \in \mathbb{Q}(C, A_2, A_3, \dots, A_n, D, B_2, B_3, \dots, B_n)$  are algebraically independent.

First, note that only  $W_{n-1}$  depends on  $D$  and  $\hat{W}_{n-1}$  only depends on  $A_2$ . Therefore, it is enough to prove that the  $2n-2$  other rational fractions are algebraically independent over

$\mathbb{Q}(A_2)$ . For the calculation, it is convenient to successively make the following two changes of variables. First, we set

$$\begin{aligned} M_k &:= 1 - \sum_{i=1}^k A_i \text{ for } 2 \leq k \leq n+1, \\ N_k &:= A_1 + \sum_{i=2}^k B_i \text{ for } 1 \leq k \leq n. \end{aligned}$$

Note that  $M_{n+1} = 0$  and  $N_1 = A_1$ . We get the formulae

$$\begin{aligned} \hat{W}_0 &= \frac{M_n}{N_{n-1}}, \\ W_{n-k} &= \frac{CM_k}{N_k} \text{ for } 2 \leq k \leq n, \\ \hat{W}_{n-k} &= 1 + \frac{M_{k+1} - N_{k-1}}{N_k} \text{ for } 2 \leq k \leq n-1. \end{aligned}$$

Then, we set

$$U_k := \frac{M_k}{N_k} \text{ and } V_k := \frac{M_{k+1}}{N_k} \text{ for } 2 \leq k \leq n,$$

and  $V_1 = \frac{1-2A_2}{A_2}$  getting the formulae

$$\begin{aligned} \hat{W}_0 &= V_{n-1}, \\ W_{n-k} &= CU_k \text{ for } 2 \leq k \leq n, \\ \hat{W}_{n-k} &= 1 + V_k - \frac{U_k}{V_{k-1}} \text{ for } 2 \leq k \leq n-1. \end{aligned}$$

Hence, the  $2n-2$  independent parameters  $C, A_3, \dots, A_n, B_2, \dots, B_n$  provide the  $2n-2$  independent parameters  $C, U_2, \dots, U_n, V_2, \dots, V_{n-1}$ . Thus, it is sufficient to show that the rational fractions  $W_0, \dots, W_{n-2}, \hat{W}_0, \dots, \hat{W}_{n-2} \in \mathbb{Q}(A_2)(C, U_2, U_3, \dots, U_n, V_2, V_3, \dots, V_{n-1})$  are algebraically independent over  $\mathbb{Q}(A_2)$ .

Suppose that there exists an irreducible polynomial  $R \in \mathbb{Q}(A_2)[X_1, \dots, X_{2n-2}]$  such that

$$R(\hat{W}_0, \dots, \hat{W}_{n-2}, W_0, \dots, W_{n-2}) = 0.$$

Specializing  $C$  in 1, we obtain

$$R\left(V_{n-1}, V_{n-1} + 1 - \frac{U_{n-1}}{V_{n-2}}, \dots, V_2 + 1 - \frac{U_2}{V_1}, U_n, \dots, U_2\right) = 0 \quad (4)$$

where the  $2n-3$  last rational fractions generate the field  $\mathbb{Q}(A_2)(U_2, \dots, U_n, V_2, \dots, V_{n-1})$  over  $\mathbb{Q}(A_2)$ . Therefore, they are algebraically independent. We investigate their relation with the first coordinate, that will provide information on  $R$ . Observe that for  $2 \leq k \leq n-1$ ,

$$\hat{W}_{n-k} = 1 + V_k - \frac{U_k}{V_{k-1}}$$

provide the relation

$$V_k = \hat{W}_{n-k} - 1 + \frac{W_{n-k}}{V_{k-1}}.$$

Since  $\hat{W}_0 = V_{n-1}$ , we can compute by finite induction

$$\hat{W}_0 = V_{n-1} = (\hat{W}_1 - 1) + \frac{W_1}{V_{n-2}} = f_0 + \mathop{\mathrm{K}}_{k=1}^{n-2} \frac{e_k}{f_k}$$

where

$$\begin{cases} e_k &= W_k & \text{for } 1 \leq k \leq n-2 \\ f_k &= \hat{W}_{k+1} - 1 & \text{for } 0 \leq k \leq n-3 \\ f_{n-2} &= V_1 = \frac{1-2A_2}{A_2} \end{cases}$$

and

$$f_0 + \mathop{\mathrm{K}}_{k=1}^{n-2} \frac{e_k}{f_k} = f_0 + \frac{e_1}{f_1 + \frac{e_2}{f_2 + \frac{\ddots}{f_{n-2}}}}$$

is Gauss' notation for a (finite) generalized continued fraction. Denote by  $\left(\frac{E_k}{F_k}\right)_{k=0}^{n-2}$  the finite sequence of its convergents.

We set

$$\tilde{R} = F_{n-2}\hat{W}_0 - E_{n-2}$$

where  $F_{n-2}$  and  $E_{n-2}$  are seen as polynomials in  $\mathbb{Q}(A_2)[W_0, \dots, W_{n-2}, \hat{W}_0, \dots, \hat{W}_{n-2}]$ . Note that  $F_{n-2}$  and  $E_{n-2}$  do not depend on  $\hat{W}_0$  since none of the  $(e_k)_{1 \leq k \leq n-2}$  and  $(f_k)_{0 \leq k \leq n-2}$  do. Hence,  $\tilde{R}$  is a polynomial of degree 1 with respect to  $\hat{W}_0$ . Writing the Euclidean division of  $R$  by  $\tilde{R}$  in  $\mathbb{Q}(A_2, \hat{W}_1, \dots, \hat{W}_{n-2}, W_0, \dots, W_{n-2})[\hat{W}_0]$  we get

$$R = \tilde{R}Q + P$$

with  $\deg_{\hat{W}_0}(P) = 0$ . Hence  $P$  can be seen as a polynomial in the  $2n-3$  variables  $\hat{W}_1, \dots, \hat{W}_{n-2}, W_0, \dots, W_{n-2}$  over  $\mathbb{Q}(A_2)$ . The latter are algebraically independent over  $\mathbb{Q}(A_2)$



because their specializations at  $C = 1$  are. We deduce that  $P = 0$ , and by irreducibility of  $R$ , the polynomial  $Q$  is a constant:

$$R = \alpha \left( F_{n-2} \hat{W}_0 - E_{n-2} \right)$$

with  $\alpha \in \mathbb{Q}(A_2)$ .

Specializing  $C$  in 0, we obtain

$$R \left( V_{n-1}, V_{n-1} + 1 - \frac{U_{n-1}}{V_{n-2}}, \dots, V_2 + 1 - \frac{U_2}{V_1}, 0, \dots, 0 \right) = 0$$

where the  $n-1$  non zero rational fractions generate the field  $\mathbb{Q}(V_1)(U_3, \dots, U_{n-1})(V_{n-1}, V_{n-2}, \dots, V_2, U_2)$  over  $\mathbb{Q}(V_1)(U_3, \dots, U_{n-1})$ . Therefore, they are algebraically independent over  $\mathbb{Q}(A_2) = \mathbb{Q}(V_1)$ . We deduce that the constant monomial of  $R$  seen in  $\mathbb{Q}(A_2, \hat{W}_0, \dots, \hat{W}_{n-2})[W_0, \dots, W_{n-2}]$  should be zero.

We now compute the constant monomial of  $F_{n-2} \hat{W}_0 - E_{n-2}$  seen in  $\mathbb{Q}(A_2, \hat{W}_0, \dots, \hat{W}_{n-2})[W_0, \dots, W_{n-2}]$ . We use the classical recurrence formulae for the convergents

$$E_{k+1} = e_{k+1} E_k + f_{k+1} E_{k-1} \text{ and } F_{k+1} = e_{k+1} F_k + f_{k+1} F_{k-1}$$

to compute the constant term of  $E_{n-2}$  and  $F_{n-2}$  to be

$$\prod_{k=0}^{n-2} f_k \text{ and } \prod_{k=1}^{n-2} f_k$$

respectively. Thus the constant monomial of  $F_{n-2} \hat{W}_0 - E_{n-2}$  seen in  $\mathbb{Q}(A_2, \hat{W}_0, \dots, \hat{W}_{n-2})[W_0, \dots, W_{n-2}]$  is

$$\left( \prod_{k=1}^{n-2} f_k \right) \hat{W}_0 - \prod_{k=0}^{n-2} f_k = (\hat{W}_0 - \hat{W}_1 + 1) \frac{1 - 2A_2}{A_2} \prod_{k=1}^{n-3} (\hat{W}_{k+1} - 1).$$

The fact that  $\hat{W}_{k+1} \neq 1$  and  $\hat{W}_0 + 1 \neq \hat{W}_1$  induces that this constant monomial is non zero. Hence  $\alpha$  and  $R$  are zero.

This proves the algebraic independence of the  $2n$  exponents. □

## References

- [1] Yann Bugeaud and Michel Laurent. On exponents of homogeneous and inhomogeneous diophantine approximation. *Moscow Math. J.*, 5:747–766, 2005.
- [2] Yann Bugeaud and Michel Laurent. Exponents of diophantine approximation. *Diophantine Geometry Proceedings*, 4:101–121, 2007.

- [3] Vojtěch Jarník. Zum khintchineschen "Übertragungssatz". *Trav. Inst. Math. Tbilissi*, 3:193–212, 1938.
- [4] Alexander Ya. Khinchin. Über eine klasse linearer diophantischer approximationen. *Rend. Circ. Mat. Palermo* 50, pages 170–195, 1926.
- [5] Alexander Ya. Khinchin. Zur metrischen theorie der diophantischen approximationen. *Math.Z.*, 24:706–714, 1926.
- [6] Michel Laurent. Exponents of diophantine approximation in dimension two. *Canad. J. Math.*, 61:165–189, 2009.
- [7] Michel Laurent. On transfer inequalities in Diophantine approximation. In *Analytic number theory*, pages 306–314. Cambridge Univ. Press, Cambridge, 2009.
- [8] Antoine Marnat. About Jarník's type relation in higher dimension. *Annales de l'Institut Fourier*, to appear.
- [9] Hermann Minkowski. *Geometrie der Zahlen*. Bibliotheca Mathematica Teubneriana, Band 40. Johnson Reprint Corp., New York-London, 1968.
- [10] Damien Roy. On Schmidt and Summerer parametric geometry of numbers. *Ann. of Math.*, 182:739–786, 2015.
- [11] Damien Roy. Spectrum of the exponents of best rational approximation. *Math. Z.*, 283:143–155, 2016.
- [12] Wolfgang M. Schmidt. On heights of algebraic subspaces and diophantine approximations. *Ann. of Math. (2)*, 85:430–472, 1967.
- [13] Wolfgang M. Schmidt and Leonhard Summerer. Parametric geometry of numbers and applications. *Acta Arithmetica*, 140(1):67–91, 2009.
- [14] Wolfgang M. Schmidt and Leonhard Summerer. Diophantine approximation and parametric geometry of numbers. *Monatsch. Math.*, 169(1):51–104, 2013.